ON THE GENERALIZED SOLVABILITY OF AN ANALOGUE OF THE FRANKL PROBLEM FOR A FOURTH-ORDER ELLIPTIC-PARABOLIC EQUATION WITH SMOOTH COEFFICIENTS

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Abstract

In this article, we study a Frankl-type problem for fourth-order mixed equations. Using the methods of functional analysis, the Lax-Milgram theorem, we prove a unique generalized solvability in Sobolev spaces with a weight of an analogue of the Frankl problem for a fourth-order mixed-type equation with smooth coefficients.

The main result of the work is the confirmation of the existence of a solution in the space $H_{2,k}$ of this mixed fourth-order equation under certain conditions related to the coefficients.

Keywords: Frankl problem, Sobolev space, norm, Lax-Milgram theorem, Hölder's inequality.

In this article, using the methods of the functional analysis with the help of the Lax-Milgram theorem, we prove a unique generalized solvability in Sobolev spaces with a weight of an analogue of the Frankl problem for a fourth-order mixed-type equation with smooth coefficients (Базаров D., Солтанов Х., 1995; Мередов М., Базаров Д., 1973; Soltanow H., 2014; Soltanow H., 2016; Франкль Ф.Н., 1973).

Let's consider the equation:

$$Lu \equiv U_{xxxx} + 2\sqrt{K(y)}U_{xxyy} + K(y)U_{yyyy} + a(x,y)U_{xxy} + a_1(x,y)U_{xyy} + b(x,y)U_{xx} +$$
$$+b_1(x,y)U_{xy} + b_2(x,y)U_{yy} + c_1(x,y)U_x + c_2(x,y)U_y + c(x,y)U = f(x,y), \tag{1}$$

where

$$K(y) = \left(\frac{1+sgny}{2}\right)y^2 K_1(y),$$
$$K_1(0) \neq 0,\ K_1(y) \in C^4_{[0,1]},\quad K_1(y) > 0.$$

is in a bounded simply connected domain D with boundary Γ, which consists of a smooth curve σ with the ends at the points $A(0,1)$, $B(1,0)$ and line segments $AA_1 : x = 0,\ A_1C : y = -1,\ CB : x = 1.$

Problem. Find a solution to the equation (1) in the domain D satisfying the condition

$$U(x,y)\big|_\Gamma = 0,\qquad U_n\big|_\sigma = U_x n_1 + U_y n_2\big|_\sigma = 0,\qquad U_x\big|_{BC} = 0,$$
$$\left[U_x(0,y) + U_x(0,-y)\right]_{AA_1} = 0,$$
$$\left[U_{xx}(0,y) - U_{xx}(0,-y)\right]_{AA_1} = 0 \tag{2}$$

where $\vec{n} = (n_1, n_2)$ is the internal normal vector.

Along with the boundary value problem (1), (2), let's consider the boundary value problem for an equation that is formally conjugate (in the sense of Lagrange) with the equation (1), i.e. for the equation

$$L^{*}V \equiv V_{xxxx} + 2\sqrt{K(y)}V_{xxyy} + K(y)V_{yyyy} + 4K'(y)V_{yyy} + \left(2\frac{K'(y)}{\sqrt{K(y)}} - a(x,y)\right)V_{xxy} - a_1(x,y)V_{yyx} +$$

$$+\left(b(x,y) - a_y(x,y) + \frac{2K(y)K''(y) - K'^2(y)}{2K^{\frac{3}{2}}(y)}\right)V_{xx} + \left(b_1(x,y) - 2a_x(x,y) - 2a_{1y}(x,y)\right)V_{xy} +$$

$$+\left(6K''(y) - a_{1x}(x,y) + b_2(x,y)\right)V_{yy} + \left(2b_x(x,y) + b_{1y}(x,y) - c_2(x,y)\right)V_y + \left(K^{'v}(y) - a_{yxx} -\right.$$

$$-c_1(x,y) - 2a_{yx}(x,y) - a_{1yy}(x,y)\Big)V_x + \left(4K'''(y) - a_{xx}(x,y) - 2a_{1yx}(x,y) + b_{1x}(x,y) + 2b_{2y}(x,y) -\right.$$

$$\left.-a_{1yyx}(x,y) + b_{xx}(x,y) + b_{1xy}(x,y) + b_{2yy}(x,y) - c_1(x,y) - c_2(x,y) + c(x,y)\right)V(x,y) = g(x,y) \qquad (3)$$

Associated problem. Find a solution $V(x,y)$ to the equation (3) in the domain D that satisfies the boundary conditions

$$V(x,y)\big|_{\Gamma} = 0,\ \ V\big|_{n\sigma} = 0,\ \ V\big|_{xBC} = 0,$$

$$\left[V_x(0,y) + V_x(0,-y)\right]_{AA_1} = 0,$$

$$\left[V_{xx}(0,y) - V_{xx}(0,-y)\right]_{AA_1} = 0 \qquad (4)$$

Let's assume that the coefficients of the operator LU are sufficiently smooth functions.

Let's introduce the necessary sets of functions. Let's denote by Φ_L the class of the smooth functions $U(x,y) \in W_2^4(D)$ satisfying, respectively, conditions (2). Let's denote by $H_{2,k}$ Sobolev space (Соболев С.А., 1962), the resulting closure Φ_L, in the norm

$$\|U\|_{2,K}^2 = \int_D \left(U_{xx}^2 + 2\sqrt{K(y)}U_{xy}^2 + K(y)U_{yy}^2\right)dD + \|U\|_1^2,$$

where $\|\cdot\|$ is the norm in the Sobolev space W_2^1. Let's introduce a bilinear form $B(U,V)$ over the space $H_{2,k} \times H_{2,k}$.

$$B(U,V) = \int_D [U_{xx}V_{xx} + 2\sqrt{K(y)}U_{xy}V_{xy} + K(y)U_{yy}V_{yy} + \frac{K'(y)}{\sqrt{K(y)}}U_yV_{xx} - K'''(y)U_yV -$$

$$-K''(y)U_yV_y + K(y)U_{yy}V_y + K'(y)U_{yy}V_y - aU_{xx}V_y - a_yU_{xx}V - a_1U_{xy}V_y + a_{1yy}U_xV_y -$$

$$-bU_xV_x - b_xU_xV - b_1U_yV_x - b_{1x}U_yV - b_2U_yV_y - b_{2y}U_yV + C_1U_xV + C_2U_yV + CUV\big]dD. \qquad (5)$$

Definition. The function $U(x,y) \in H_{2,k}$ will be called generalized solution to the problem (1), (3) if for all function $V(x,y) \in \Phi_L$ we have the identity

$$\left(U, L^{*}V\right)_{\circ} = B(U,V) = (f,V)_{\circ} \equiv \int_D fVdD$$

The main result of this work is the assertion that, under certain conditions on the coefficients of equation (1), there exists a solution from the space $H_{2,k}$.

Lemma 1. Let in the domain D the following conditions are performed

$$A_{11} > \delta > 0,\ \ A_{13} > \delta > 0,$$

$$A_{12} \le A_{11} \cdot A_{13},\qquad A_{14} > \delta > 0, \qquad (6)$$

where

$$A_{11}=\frac{1}{2}a_y-b-\frac{2KK''-K'^2}{4K^{\frac{3}{2}}},\ 2A_{12}=a_x+a_{1y}-b(x,y),\ A_{13}=-2K''(y)+\frac{1}{2}a_{1x}-b_2(x,y),$$

$$A_{14}=-\frac{1}{2}K^{('V)}(y)-\frac{1}{2}a_{xxy}-\frac{1}{2}a_{1xyy}+\frac{1}{2}b_{xx}+\frac{1}{2}b_{1yx}+\frac{1}{2}b_{2yy}-\frac{1}{2}C_{1x}-\frac{1}{2}C_{2y}+\mathrm{C}.$$

Then for any function $U(x,y)\in\Phi_L$ there is an inequality

$$\left|(Lu,U)\right|\equiv B(U,U)\geq m\|U\|_1^2+\int_D\left[U_{xx}^2+2\sqrt{K(y)}U_{xy}^2+K(y)U_{yy}^2\right]dD,\ \ m>0. \tag{7}$$

Proof. In the domain D we consider the integral $(LU,U)_\circ$ Since the function $U(x,y)$ satisfies the boundary condition (2), integrating by parts, we obtain

$$(LU,U)_\circ=\int_D\left(A_{11}U_x^2+2A_{12}U_xU_y+A_{13}U_y^2+A_{14}U^2+U_{xx}^2+2\sqrt{K(y)}U_{xy}^2+K(y)U_{yy}^2\right)dD+\int_{-1}^{1}U_xU_{xx}dy$$

It is not difficult to show that

$$\int_{-1}^{1}U_xU_{xx}dy\,|_{x=0}=0$$

Indeed,

$$\int_{-1}^{1}U_xU_{xx}\,|_{x=0}=\int_{-1}^{\circ}U_xU_{xx}dy\,|_{x=0}+\int_{\circ}^{1}U_xU_{xx}dy\,|_{x=0}=\int_{\circ}^{1}U_x(0,-y)U_{xx}(0,-y)\,dy+$$

$$+\int_{\circ}^{1}U_x(0,y)U_{xx}(0,y)\,dy=-\int_{\circ}^{1}U_x(0,y)U_{xx}(0,y)\,dy+\int_{\circ}^{1}U_x(0,y)U_{xx}(0,y)\,dy=0.$$

Therefore,

$$\left|(Lu,U)\right|\geq m\int_D\left(U_x^2+U_y^2+U^2\right)dD+\int_D\left(U_{xx}^2+2\sqrt{K(y)}U_{xy}^2+K(y)U_{yy}^2\right)dD.$$

This implies the assertion of Lemma 1, which implies the uniqueness of the solution to the stated problem from the class Φ_L.

Lemma 2. Let the condition

$$a_1\leq M\sqrt{K} \tag{8}$$

is performed. Then the bilinear $B(U,V)$ is continuous over the spaces $H_{2,k}\times H_{2,k}$ i.e. the estimation

$$\left|B(U,V)\right|\leq M\,\|U\|_{H_{2,k}}\cdot\|V\|_{H_{2,k}},\ U,V\in H_{2,k} \tag{9}$$

is performed.

Proof. Using Hölder's inequality and estimate (8) for the coefficient $a_1(x,y)$ from (5) we obtain (9).

Theorem 1. Let conditions (6), (8) be satisfied. Then for any function $f\in L_2$ there exists and, moreover, a unique generalized solution to the problem (1), (2) from the space $H_{2,k}$.

The proof: as it is known (Нагумо М., 1967), for the solvability of the functional equation

$$B(U,V)=(f,V)_\circ,\qquad \forall V\in\Phi_L$$

it is sufficient to condition that the bilinear form $B(U,V)$ was continuous, i.e.

$$\left|B(U,V)\right|\leq M\|U\|_{H_{2,k}}\cdot\|V\|_{H_{2,k}}$$

and the inequality

$$\left|B\left(U,U\right)\right| \geq m\left\|U\right\|_{H_{2,k}}^{2}$$

is performed.

And these inequalities were proved in Lemmas 1, 2.

Hence, the boundary value problem (1), (2) is uniquely solvable in the space $H_{2,k}$.

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